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Quantum Theory of the Hall Effect

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We discuss a model of both the classical and the integer quantum Hall effect which is based on a semiclassical Schrödinger–Chern–Simons action, where the Ohm equations result as equations of motion. The quantization of the classical Chern–Simons part of action under typical quantum Hall conditions results in the quantized Hall conductivity. We show further that the classical Hall effect is described by a theory which arises as the classical limit of a theory of the quantum Hall effect. The model also explains the preference and the domain of the edge currents on the boundary of samples.

1. INTRODUCTION AND SUMMARY

Recently, we discussed a model of the integer quantum Hall effect (IQHE)[2] according to which the quantization of the Hall conductivity should result from the quantum electrodynamics in 2+1 dimensions (Ghaboussi, n.d.-a,b). In this semiclassical Schrödinger–Chern–Simons model the Hall conductivity σ_H appears as the normalization parameter of the Chern–Simons action.[3] Furthermore, we assumed there, in accord with the experimental results on the QHE, a vanishing longitudinal conductivity σ_L (see footnote 2). Then the Ohm equations of IQHE with quantized σ_H are obtained as

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[2] For a general review of the quantum Hall effect and its experimental setting see Prange and Girvin (1987), Macdonald (1989), Morandi (1992), and Janssen *et al.* (1994).

[3] The use of σ_H as a normalization parameter of the Chern–Simons action is in accordance with the use of similar parameters in interacting systems of particles which then become proportional to σ_H in FQHE models (Semenof, 1988; Zhang, 1989). Recall that it is expected that noninteracting particles in quantum Hall samples result under proper conditions in the IQHE, whereas the interacting particle systems should be responsible for the fractional QHE (FQHE). In the last case it seems that, depending on the theoretical treatment of the question of the ground state, one is led to one of the above-mentioned models.

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the equations of motion from the Schrödinger–Chern–Simons action with quantized electromagnetic potentials (Ghaboussi, n.d.-a,b).

Here we discuss a more general model for both the classical Hall effect (CHE) and the IQHE, where the related Ohm equations result as equations of motion also from a Schrödinger–Chern–Simons action functional. Then the quantum Hall conditions (von Klitzing, 1995; Knott *et al.*, 1995) cause the transition of the Hall system to the quantum regime, where the necessary quantization of the electromagnetic potentials results in the quantized σ_H in the absence of σ_L. It is a model of noninteracting charge carriers for the IQHE with a semiclassical Schrödinger–Chern–Simons action functional; hence we quantize not the Schrödinger term, which represents the charge carrier system, but only the Chern–Simons term, which represents the dynamics of the almost pure gauge potentials (Ghaboussi, n.d.-a,b). Thus, a second quantization of the Schrödinger term in our model which corresponds to the interacting particle system should result after solution of the ground state in a FQHE model similar to known models (see footnote 3).

Our model is based on the following standpoint on the theory of Hall effects: that because there are both CHE and QHE (IQHE and FQHE), the theory of the QHE must be the quantization of the "classical" theory of the CHE (Ghaboussi, n.d.-a, b). Furthermore, rigorous quantization of a system requires the knowledge of its action functional. Accordingly, we have to construct first a "classical" action for the CHE, from which the resulting equations of motion must explain the CHE behavior. On the other hand the "classical" Ohm equations (see footnote 2) are the only equations which describe the CHE. Thus, the action which should describe the CHE has to result in the Ohm equations as its equations of motion. This interpretation of the Ohm equations as the equations of motion which must result directly from an action functional is a new element of our standpoint. In all other models the Ohm equations are considered as a given relation in the sense of "material" or "phenomenological" equations.[4]

On the other hand, in view of the well-known fact that these Ohm equations are semiclassical relations with Schrödinger-type current densities for electrons, the desired action for CHE also should be of the semiclassical type as in our model. Then, the canonical quantization of the classical part of this action for the case of noninteracting electrons must result in the quantum theory of the IQHE and also in the quantized Hall conductivity according to the IQHE.

To investigate the relation between the QHE and the CHE, let us analyze first the Ohm equations for the QHE and the CHE These are given by

[4]For a different model, where only the Ohm equations of the IQHE, but not of the CHE, are derived as equations of motion, see Fröhlich and Kerler (1991).

$$j_m = \sigma_H \epsilon_{nm} E_n, \qquad \epsilon_{mn} = -\epsilon_{nm} = 1; \qquad m, n = 1, 2 \tag{1}$$

for the QHE, where $\sigma_H = en/B$ becomes quantized in units of e^2/h. Here n is the global surface density of the charge carriers ("electrons") and $B := B_3$ is the applied magnetic field.[5] On the other hand, the Ohm equations for the CHE are given by

$$j_m = \sigma_H \epsilon_{nm} E_n + \sigma_L E_m \tag{2}$$

with $\sigma_L = \sigma_0/[1 + (\omega_c\tau)^2]$ and $\sigma_H = \sigma_0(\omega_c\tau)/[1 + (\omega_c\tau)^2]$, where $\sigma_0 = e^2n\tau/\mu$, $\omega_c := eB/\mu$, and τ and μ are, respectively, the mean free time and the mass of electrons.[6]

The key observation is that according to quantum mechanics (Landau and Lifschitz, 1976), the current density of electrons in a *magnetic field* without spin term and with $C = 1$ is given by (a) $j_m := (ie\hbar/2\mu)[(\partial_m\psi^*)\psi - \psi^*(\partial_m\psi)] - (e^2/\mu)A_m\psi^*\psi$, whereas the current density of electrons in the limit $B \to 0$, i.e., for $\omega_c\tau << 1$, should be given by (b) $j_m := (ie\hbar/2\mu)[(\partial_m\psi^*)\psi - \psi^*(\partial_m\psi)]$, both obeying the continuity equation (Landau and Lifschitz, 1976). We deduce that relation (a) is valid in the integer quantum Hall regime ($\omega_c\tau >> 1$), where the external magnetic field is large, whereas relation (b) is valid in the classical Hall regime ($\omega_c\tau << 1$), where the same external field is small or absent.

The semiclassical Schrödinger–Chern–Simons action functional in 2 + 1 dimensions is the only action from which we can obtain the mentioned Ohm equations (1) and (2) as the equations of motion (see below) (Ghaboussi, n.d.-a,b), where σ_H plays the role of the normalization parameter of the classical Chern–Simons action. To see the relation of the quantization of the Hall system with the empirical quantum behavior under typical quantum Hall conditions (von Klitzing, 1995; Knott *et al.*, 1995), let us recall that in a strong magnetic field the Hall conductivity σ_H becomes *small* according to its definition as given above. Precisely, in the quantum Hall limit, i.e., $\omega_c\tau >> 1$, the σ_H and σ_L should be considered, according to their definitions given above, of the order $(\omega_c\tau)^{-1}$ and $(\omega_c\tau)^{-2}$ respectively, i.e., $\sigma_H << 1$

[5] Precisely, the total magnetic field acting on the Hall system described by the Schrödinger–Chern–Simons action (9) is given by $B_{total} := B_{external} + B(A_m)$, with $B_{external} >> B(A_m)$, where $B_{external}$ is the external homogeneous strong magnetic field applied to the system. The $B(A_m)$ is the magnetic field arising from the dynamics of A_m potentials, which is also responsible for the electric fields E_m. The $B(A_m)$ is usually so small that $\omega_c\tau << 1$ and so its influence on the conductivity is contained already in what is known under the classical Hall effect, since, to achieve the magnetic influence of the quantum Hall type one needs strong magnetic fields, such as those used in QHE experiments [see von Klitzing (1995), Knott *et al.* (1995), and references in footnote 2].

[6] Recall also that relation (2) can be obtained from relation (1) by an infinitesimal $SO(2)$ transformation in the E_m or in the A_m space. The infinitesimal angle $\delta\chi = \sigma_L/\sigma_H$ becomes almost zero in the quantum Hall regime. See also footnote 2.

and $\sigma_L << \sigma_H$ or $\sigma_L \to 0$. Moreover, in this limit the Hall conductivity is given by $\sigma_H = ne/B$, so that for small n and large B_{external} the σ_H becomes small. Thus, if we consider in our model σ_H as the normalization parameter of the Chern–Simons action S_{CS} (see footnote 3) and quantize this action according to the Schrödinger representation (Ghaboussi, n.d.-a,b)

$$\Psi_{CS}(A) \propto e^{i\sigma_H S_{CS}/\hbar} \tag{3}$$

then $\sigma_H S_{CS}$ also becomes small for relevant S_{CS} actions in view of the above-mentioned smallness of σ_H. Therefore, for small $\sigma_H S_{CS}$, i.e. precisely for those $\sigma_H S_{CS}$ which are comparable with $\hbar$, the quantum behavior of the action becomes dominant (Feynman and Hibbs, 1965) and we encounter the integer quantum Hall regime manifested by the IQHE. Moreover, in this quantum limit the σ_L becomes, as mentioned above, *very small*, tending to zero, as expected in the QHE.

Conversely, if the magnetic field is not strong, i.e. for $\omega_c\tau << 1$, σ_H and $\sigma_H S_{CS}$ become large or $\sigma_H S_{CS} >> \hbar$ and we have the classical regime, where the quantum fluctuations of the action are compensated (Feynman and Hibbs, 1965) and the original quantum theory reduces to its classical limit, which is the theory of the CHE. In this classical limit $\sigma_L \approx \sigma_0$; thus both types of conductivity are no longer small, but have considerable magnitudes, since they are also present in the Ohm equations of the CHE (2). We neglect here the typical FQH conditions including the high mobility of particles in view of the fact that we consider only the IQHE.

On the other hand, it is known that if one considers currents involved in the IQHE only as boundary currents, then most of experimental data can be understood in a satisfactory manner (von Klitzing, 1995; Knott *et al.*, 1995). It is a feature of the Chern–Simons ansatz in a manifold with a spatial boundary that the boundary currents are the only allowed ones according to the constraints of the theory under typical quantum Hall conditions. Therefore, for construction of a theoretical model for both the CHE and the IQHE one is left with the Schrödinger–Chern–Simons action, from which we know already that it results, at least, in the Ohm equations for the CHE and the IQHE as the equations of motion (Ghaboussi, n.d.-a,b).

2. THE CHERN–SIMONS ACTION FOR CLASSICAL AND QUANTUM HALL EFFECT

The general action from which the Ohm equations of the CHE and the IQHE [(2) and (1)] can be obtained as the equations of motion is the following

Schrödinger–Chern–Simons action defined on the (2 + 1)-dimensional manifold $M = \Sigma \times \mathbf{R}$:

$$S = \frac{1}{8\pi}\int dt \int_{\Sigma} \psi^*\left[i\hbar\partial_t - \frac{1}{2\mu}(-i\hbar\partial_m - eA_m)^2 - eA_0\right]\psi + \text{h.c.}$$

$$- \frac{\sigma_H}{8\pi}\int_M \epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma \tag{4}$$

where $A_\alpha(x_m, t)$ is still the classical electromagnetic potential, which remains classical in the classical Hall regime but must be quantized in the quantum Hall regime. Furthermore, $\{\alpha, \beta, \gamma\} = \{0, 1, 2\}$ and everywhere $C = 1$ and $\partial_m = \partial/\partial x_m$ and we consider (in accordance with the experimental arrangements of the QHE) that Σ has a boundary. Furthermore, as already mentioned, the Schrödinger term represents the mechanics of the noninteracting particle system, whereas the Chern–Simons term represents the dynamics of the coupled electromagnetic potentials.

Obviously, we use the σ_H as the locally constant normalization parameter of the Chern–Simons action. We are justified because σ_H can be considered as a dimensionless and locally constant quantity in 2 + 1 dimensions also in view of its well-known topological or global character.[7] Moreover, we suppressed the spin term within the usual Schrödinger action for an "electron" in a magnetic field in view of the well-known fact about the QHE that spin degeneracy is not essential for the IQHE (see footnote 2).

In view of the gauge freedom of A_m we choose the gauge-fixing condition $A_0 = 0$ to retain the true degrees of freedom of the electromagnetic fields in the action (4). Then the action reduces to

$$\frac{1}{8\pi}\int dt \int_{\Sigma} \psi^*\left[i\hbar\partial_t - \frac{1}{2\mu}(-i\hbar\partial_m - eA_m)^2\right]\psi + \text{h.c.} \tag{5}$$

$$- \frac{\sigma_H}{8\pi}\int dt \int_{\Sigma} \epsilon^{mn}\dot{A}_m A_n$$

The equations of motion for classical A_m potentials which result from this action are

$$j_m - \frac{e^2 n}{\mu} A_m = \sigma_H \epsilon^{nm}\dot{A}_n \tag{6}$$

[7]This means that $d\sigma_H = 0$. Furthermore, recall also that both charge carrier density n in two dimensions and the B field are of dimension L^{-2}. Thus, $\sigma_H = en/B$ becomes dimensionless. For further arguments in favor of the local constancy of σ_H see Frölich and Kerler (1991). Also see footnotes 1 and 3.

where we used according to $\omega_c\tau << 1$ in the classical regime the corresponding definition (b) $j_m := (ie\hbar/2\mu)[(\partial_m\psi^*)\psi - \psi^*(\partial_m\psi)]$.

We introduce the gauge $A_m = E_m\tau$ in (6), which is more appropriate for the case of low magnetic fields, i.e., precisely it is appropriate for the classical Hall regime with $\omega_c\tau << 1$.[8] It is equivalent to the relaxation-time approximation, which is the usual approach in this case (Callaway, 1964).[9] Substituting $A_m = E_m\tau$ in (6), we obtain the desired Ohm equations for the CHE

$$j_m = \sigma_L E_m + \sigma_H \epsilon_{nm} E_n \tag{7}$$

where we used $\sigma_L \approx \sigma_0$ according to $\omega_c\tau << 1$.

Thus, we obtained the Ohm equations of the CHE as the equations of motion from the action (4) in the classical Hall regime consistently, according to $\omega_c\tau << 1$.

The quantization of the action (5) under typical IQH conditions (von Klitzing, 1995; Knott *et al.*, 1995), i.e., in the limit $\omega_c\tau >> 1$, results then in the action which is responsible for the Ohm equations of the IQHE, where one must use obviously the definition (a) for the current density in the quantum Hall regime according to $\omega_c\tau >> 1$.

Recalling our previous analysis, we mention that the quantum regime of the Hall effect is related in a double sense to the strong exterior magnetic field which is applied to two-dimensional electronic systems: In the limit $\omega_c\tau >> 1$ the σ_H and σ_L should be considered theoretically of the order $(\omega_c\tau)^{-1}$ and $(\omega_c\tau)^{-2}$, respectively, i.e., σ_H becomes small and σ_L tends to zero, as confirmed by experiments (von Klitzing, 1995; Knott *et al.*, 1995). On the other hand, under typical quantum Hall conditions where the number or the density of electrons is small the σ_H and $\sigma_H S_{CS}$ become smaller and so the latter becomes comparable with $\hbar$, which results in the integer quantization of σ_H, as also confirmed by experiments (von Klitzing, 1995; Knott *et al.*, 1995).

In other words, the $\omega_c\tau >> 1$ limit together with small n corresponds to the quantum regime (von Klitzing, 1995; Knott *et al.*, 1995), where $\sigma_H S_{CS}$ becomes comparable with $\hbar$, whereas the $\omega_c\tau << 1$ limit together with n around the usual electronic density in metals corresponds to the classical limit, where the action $\sigma_H S_{CS} >> \hbar$. Therefore, for large magnetic fields and small density of electrons, which are the typical quantum Hall conditions, the two-dimensional Hall system is in the IQHE regime (von Klitzing, 1995;

[8] Recall that in the presence of magnetic fields the well-known Landau gauge is given by $A_m = Bx_n\epsilon_{mn}$ (Landau and Lifschitz, 1976).

[9] Recall also that the relaxation time τ is indeed introduced in this approximation to calculate the electric conductivity from the Ohm equations. One can show that the usual relaxation-time approximation results in the approximation $\Delta A_m = E_m\Delta t \approx E_m\tau$, where the defining vanishing average velocity $\overline{V} = 0$ is given according to the operator $\hat{V}_m = \hat{P}_m - e\hat{A}_m$.

Knott *et al.*, 1995) which is described by the same action (4) or (5) after gauge fixing:

$$\frac{1}{8\pi}\int dt \int_{\Sigma} \psi^* \left[i\hbar\partial_t - \frac{1}{2\mu}(-i\hbar\partial_m - eA_m)^2 \right]\psi + \text{h.c.} \qquad (8)$$
$$- \frac{\sigma_H}{8\pi}\int dt \int_{\Sigma} \epsilon^{mn}\dot{A}_m A_n$$

but in view of $\sigma_H S_{CS} \approx \hbar$ with A_m potentials now obeying the usual quantization algebra (Witten, 1989; Jackiw, 1990; Dunne *et al.*, 1989; Dunne and Treugenberger, 1989)

$$[\hat{A}_m(x_l, t), \hat{A}_n(y_l, t)] = \frac{4\pi i\hbar}{\sigma_H}\epsilon_{mn}\delta^2(X - Y); \qquad X, Y \in \Sigma \qquad (9)$$

which can be read off directly from the Chern–Simons action in (8). This means that $\hat{A}_m := \partial\hbar i/\partial A_n$, which is the usual polarization of the $\{A_m\}$ phase space.

However, for practical use it is convenient to introduce the Schrödinger representation $\Psi(A) \propto e^{(i/\hbar)\sigma_H S_{CS}}$ of the Chern–Simons action

$$-\frac{\sigma_H}{8\pi}\int dt \int_{\Sigma} \epsilon^{mn}\dot{A}_m A_n \qquad (10)$$

after its quantization according to (9); hence $\Psi(A)$ must satisfy relation (9) in the sense of its eigenfunctions.

To obtain $\Psi(A)$, we use the method introduced in our previous work on IQHE (Ghaboussi, n.d.-a,b). It is based on the representation of the state functions $\Psi(A)$ in terms of the eigenstates of the quantum orbital angular momentum. For equivalent quantization of S_{CS} and its Schrödinger representations see Witten (1989), Jackiw (1990), Dunne *et al.* (1989), and Dunne and Treugenberger (1989).

Introducing polar coordinates in the phase space described by the action (10), we find that the quantum orbital angular momentum becomes $\hat{L} = -i\hbar\partial_\phi$ (Landau and Lifschitz, 1976). Then $\Psi(A)$ is given as the eigenstates of the operator $\hat{L}$ by

$$\Psi(A) = F(R)e^{(i/\hbar)\sigma_H l\phi} \qquad (11)$$

Here $F(R)$ is an arbitrary function of R and $l = R^2$ is the value of angular momentum of the system, which is a constant of motion according to the $SO(2)$ symmetry of the system. We normalize the constant $l = 1$.

Thus, the necessary single-valuedness of $\Psi(A)$ forces the σ_H to be

$$\sigma_H = 0, 1, 2, \ldots, N, \ldots; \qquad N \in \mathbf{Z}_+ \tag{12}$$

where we restricted consideration to positive values.[10]

Recall that the normalization parameter of the Ψ_{CS} always becomes quantized as an integer in view of the single-valuedness of Ψ_{CS} in its first quantization no matter what kind of quantization is performed (Witten, 1989; Jackiw, 1990; Dunne *et al.*, 1989; Dunne and Treugenberger, 1989).

Empirically, it is the mentioned typical IQH conditions (von Klitzing, 1995; Knott *et al.*, 1995) which prepare the electrons, according to their density and mobility and the strength of the exterior magnetic fields, to be in the IQHE situation (see also the conclusion).

The equations of motion for the A_m potentials which result from the quantized action (8) for the noninteracting system of charge carriers, according to (11)–(12) and using the corresponding definition (a) for the current density in magnetic fields, are

$$j_m = \sigma_H \epsilon_{nm} E_n \tag{13}$$

which are the desired Ohm equations with quantized σ_H.

It is obvious from the comparison between the quantized Chern–Simons action in units of $\hbar$, i.e., $\sigma_H S/\hbar$, and the Schrödinger action in (8) that in the atomic units the σ_H should be considered in units of e^2/h, which is equivalent to a redefinition of the quantized A_m potentials absorbing the coupling constant e. Thus, we have obtained the quantized Ohm equations of the IQHE as the equations of motion from the quantized Schrödinger–Chern–Simons action.

To summarize the quantum and classical behavior in this model, let us recapitulate the analysis of the integer quantum and classical Hall conditions:

If the Hall system is prepared with $\omega_c\tau >> 1$ and with small n, then the quantum modes of its action become dominant, but if it is prepared with $\omega_c\tau << 1$ and with n around the density of CHE samples, then its classical modes become dominant.

The theoretical description of this situation is in accord with our model, so that the general semiclassical action functional for both cases should be given by (4), where the Schrödinger term remains the same in both cases in view of the noninteracting particles in the IQHE. Then, the action (4) with quantized Chern–Simons term describes the integer quantum Hall regime,

[10]The fractional quantization of the normalization parameter should be a result of the multivaluedness of the wave function of the electrons in the Schrödinger term in its second quantization, which is related to the interacting electrons (see the models quoted in footnote 3). It is well known that the mentioned properties of electrons, such as mobility and also the strength of the exterior magnetic field, differ for FQHE samples (see footnote 2).

whereas the action (4) with classical Chern–Simons term describes the classical Hall regime.

In the first case the typical quantum Hall conditions, i.e., $\omega_c\tau >> 1$, and small n cause the smallness of σ_H so that $\sigma_H S_{CS}$ becomes comparable with $\hbar$. Thus the quantum modes of the action $\sigma_H S_{CS}$ which are represented by $\Psi(A)$ become dominant, requiring the quantization of σ_H, since the total quantum action results in the "quantum" Ohm equations with integrally quantized σ_H and vanishing σ_L, as shown above.

In the second case the action is of the order $\sigma_H S_{CS} >> \hbar$; therefore the classical limit of the Chern–Simons action, i.e., the classical Chern–Simons action, becomes dominant. Then the total action reduces to the semiclassical Chern–Simons–Schrödinger action, which describe the semiclassical theory of the CHE, since it results in the "classical" Ohm equations, as shown above.

Thus, in the theory of the CHE, its action arises as the classical limit from the quantum action of the IQHE.

3. THE EDGE CURRENTS IN QHE

Obviously, the motion of the system which is described by the action (8) together with the quantization relations (9)–(12) is constrained by the constraint

$$-\sigma_H \epsilon^{mn} \partial_m A_n = e\psi^* \psi \quad (14)$$

with $e\psi^*\psi := j_0$.

If we integrate the relation (14) over the sample surface and consider $B := \epsilon_{nm}\partial_m A_n$ as a constant field strength, then we obtain the well-known relation between the Hall conductivity and the magnetic field, namely

$$\sigma_H = ne/B \quad (15)$$

where $n = (a)^{-1} \int da(\psi^*\psi)$ is the global density of charge carriers and a is the sample area. Recall that the relation (15) conforms with the general definition of σ_H in the limit $\omega_c\tau >> 1$.[11]

However, the constraint (14) influences the motion of the IQHE system in a way which is known from the experimental results on the IQHE.

To see this let us note first some of the main experimental features of IQHE [as reviewed from von Klitzing (1995) and Knott *et al.* (1995)]:

1. Most of the IQHE data can be understood in a satisfactory manner if one reduces the involved currents to the edge currents.

2. The typical IQHE regime is related to very large B and small n.

[11] Recall also that σ_H becomes $\sigma_H = ne/B$ only in the quantum Hall limit, whereas in the classical Hall limit it is given by $\sigma_H = \sigma_0\omega_c\tau$.

3. Under integer quantum Hall conditions the edge of the Hall system is characterized by $n \rightarrow 0$.

4. For large current densities the IQHE cannot be simply described by edge currents located on the boundary, whereas low currents are transported by the edge channels.

All these features of the IQHE can be understood if we take into account the constraint (14). Recall that, in view of the Ohm equations, the currents are restricted to those regions where the A_m potentials are allowed to exist. Thus, the question of the edge currents is related to the question of the region where the A_m potentials are defined. Moreover, according to the constraint (14), the potential A_m becomes pure gauge potential with vanishing field strength if $n \rightarrow 0$.

This is the case if one has samples with small n and large B, for example, on the edges of a quantum Hall system. Thus, under these circumstances we should replace the constraint (14) by

$$\epsilon^{mn}\partial_m A_n \approx 0 \tag{16}$$

for systems under quantum Hall conditions (von Klitzing, 1995; Knott *et al.*, 1995). Then A_m potentials become pure gauge potentials, i.e., $A_m \approx ig^{-1}\partial_m g$, where g is an element of the $U(1)$ gauge group. Recall, however, that this is a local relation in quantum mechanics, and therefore 1. it should be valid only within the limit of the uncertainty relations and 2. a locally pure gauge potential has well-known geometric, globally well-defined and observable effects in quantum mechanics.[12]

On the other hand, the constraint tensor $\epsilon_{mn}\partial_m A_n$ generates a gauge transformation $A'_m = A_m + \partial_m \lambda$ in the phase space of the A_m potentials (Witten, 1989; Jackiw, 1990; Dunne *et al.*, 1989; Dunne and Treugenberger, 1989). Therefore, according to the constraint (16), one must identify $A'_m = A_m$ everywhere in the phase space. Furthermore, if, as in our case, Σ possesses a boundary, we must choose boundary conditions for A_m and λ on the boundary. We choose free boundary conditions for A_m, but $\lambda = 0$ on the boundary. A reason for this choice is that the Chern–Simons action is not invariant under gauge transformations that do not vanish on the boundary (Witten, 1989; Jackiw, 1990; Dunne *et al.*, 1989; Dunne and Treugenberger, 1989).

Accordingly, it must be required that $A'_m = A_m$ for any λ which vanishes on the boundary $\partial\Sigma$. The only pure A_m gauge potentials which obey this

[12] It is this discrepancy between the local and global properties of the pure gauge potential which makes a classical or even a semiclassical understanding of QHE difficult. The quantum mechanical, i.e., global or invariant, character of the pure gauge potential is given by its line integral, which is the phase of the wave function and results in the flux quantization. Furthermore, we mean here always a pure gauge potential of the electromagnetic or $U(1)$ type in a multiply connected region.

additional condition are those restricted to be defined only on the boundary (Witten, 1989; Jackiw, 1990; Dunne *et al.*, 1989; Dunne and Treugenberger, 1989). In other words, the only A_m potentials obeying both restrictions caused by the constraint (16) are restricted to exist on the boundary region of Σ. Then the currents j_m should be considered also to be restricted to the boundary region of Σ, i.e., to the so-called edge currents. Accordingly, under quantum Hall conditions (von Klitzing, 1995; Knott *et al.*, 1995), the edge currents are the preferred ones.

It is important to mention that if we consider the restrictions of the potentials and currents to the boundary or to the edge of the Hall system "quantum mechanically," then there is an uncertainty of the position of the currents, or so to say there is an uncertainty of the "quantum mechanical" edge $\Delta(\partial\Sigma)$ in view of the Heisenberg uncertainty relations. Thus, if we consider the uncertainty of momentum equal to $(2m\Delta E)^{1/2}$ with $\Delta E = E_o = \hbar\omega_c/2$, the uncertainty of the mentioned edge or the width of the currents orbit is given by $\Delta X = (\hbar/eB)^{1/2}$, which is the magnetic length l_B, since the edge current, according to its empirical definition, is the current which in the ideal case flows close to the edge within the length scale of the magnetic length (von Klitzing, 1995; Knott *et al.*, 1995). Moreover, this circumstance shows also that the constraint (16) should be satisfied within the uncertainty dictated by the energy–time uncertainty relation, since $\Delta E \propto B$ in the Landau levels (Landau and Lifschitz, 1976).

On the other hand, if n is large for large transport currents the right-hand side of the constraint (14) and thereby also the field strength in (14) are obviously nonvanishing and the IQHE breaks down, as shown by early experiments (von Klitzing, 1995; Knott *et al.*, 1995).

4. CONCLUSION

We have presented a model of IQHE based on a noninteracting system of charge carriers coupled to an electromagnetic potential in 2 + 1 dimensions. There are strong hints that the FQHE, which is believed to be a many-particle effect, i.e., to involve interacting particles, should result from the second quantization of the Schrödinger field of charge carriers involved in an action similar to the one used in this model (see footnote 3). Hence, the conformity of our model for the IQHE with an earlier model of the FQHE (see footnote 3) is a hint about the possibility that, if one considers a proper modification of our model for the case of interacting charge carriers, then after the second quantization of the Schrödinger term in our action for the interacting ("many-particle") system should arrive at a theory of the FQHE. However, this is possible if one can solve the problem of the ground state of interacting particles in such models (see footnote 3). We discuss the second quantization of our model and the resulting fractionality elsewhere (Ghaboussi, n.d.-c).

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